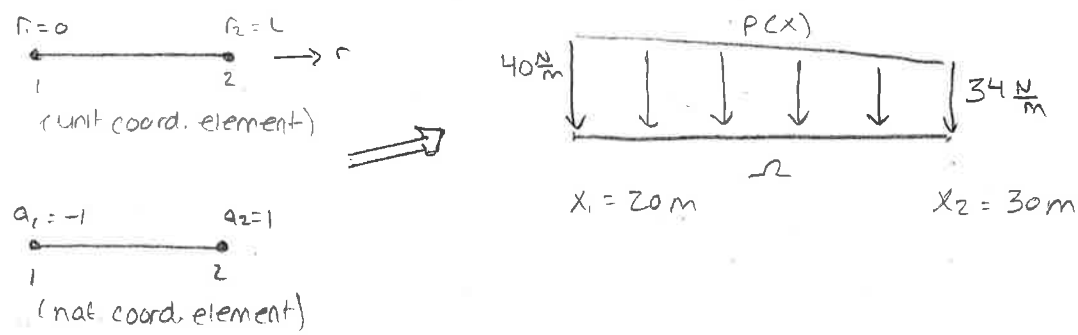
Homework #7

- Repeat "Parametric Line Pressure" solution replacing the pressure with a cubic interpolation, while leaving the geometry interpolated linearly. After the integrals are complete set the internal node pressures to linear from P_1 and P_2 before substituting the numerical values.



$r = \frac{a+1}{2}$

<u>Given</u>: A line segment from x_1 to x_2 is subjected to a pressure per unit length, $p(x)$.

i	X_i	P_i
1	20m	40 N/m
2	30m	34 N/m

Use parametric geometry to find the position, pressure value, and pressure gradient at a point 37% along the length of the line. Also, find the total force,

$$F = \int_{X_1}^{X_2} P(x)\, dx,$$

and the moment of the force with respect to the origin:

$$m = \int_{X_1}^{X_2} x\, P(x)\, dx$$

Solution # 1: Unit coordinate element + cubic interpolation

$$0 \le r \le 1$$

$$x(r) = \sum_{i=1}^{4} H_i(r)\, x_i^e = [H_1 \quad H_2 \quad H_3 \quad H_4] \begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix}$$

$$P(r) = \underline{H}(r)\, \underline{P}^e = [H_1 \quad H_2 \quad H_3 \quad H_4] \begin{Bmatrix} P_1^e \\ P_2^e \\ P_3^e \\ P_4^e \end{Bmatrix}$$

where:

$$\left.\begin{aligned} H_1 &= \frac{2 - 11r + 18r^2 - 9r^3}{2} \\ H_2 &= \frac{18r - 45r^2 + 27r^3}{2} \\ H_3 &= \frac{-9r + 36r^2 - 27r^3}{2} \\ H_4 &= \frac{2r - 9r^2 + 9r^3}{2} \end{aligned}\right\rangle \quad \Sigma_K H_K(r) \equiv 1$$

The local point of interest is $r = 0.37$

– Position $x = 0.37$

$$x(0.37) = [H_1(0.37) \quad H_2(0.37) \quad H_3(0.37) \quad H_4(0.37)] \begin{Bmatrix} 20m \\ 23.\overline{33}m \\ 26.\overline{67}m \\ 30m \end{Bmatrix} = 23.7\,m$$

(MathCAD)

– Pressure @ $x = 0.37$

$$P(0.37) = [H_1(0.37) \quad H_2(0.37) \quad H_3(0.37) \quad H_4(0.37)] \begin{Bmatrix} 40\,N/m \\ 38\,N/m \\ 36\,N/m \\ 34\,N/m \end{Bmatrix} = 37.78\,N/m$$

– Pressure gradient:

$$\frac{dP}{dx} = \frac{\partial P}{\partial r} \frac{\partial r}{\partial x}$$

$$\frac{dx}{dr} = \frac{d}{dr} \underline{H}(r) \underline{x}^e = \left[\frac{\partial H_1}{\partial r} \quad \frac{\partial H_2}{\partial r} \quad \frac{\partial H_3}{\partial r} \quad \frac{\partial H_4}{\partial r}\right] \begin{Bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{Bmatrix}$$

$$\frac{\partial H_1}{\partial r} = \frac{-11+36r-27r^2}{2}$$

$$\frac{\partial H_2}{\partial r} = \frac{18-90r+81r^2}{2}$$

$$\frac{\partial H_3}{\partial r} = \frac{-9+72r-81r^2}{2}$$

$$\frac{\partial H_4}{\partial r} = \frac{2-18r+27r^2}{2}$$

$$\frac{dx}{dr} = 10m \quad ; \quad \frac{dr}{dx} = \frac{1}{10m}$$

$$\frac{dP}{dr} = \frac{d}{dr} H(r) P^e = \begin{bmatrix} \frac{\partial H_1}{\partial r} & \frac{\partial H_2}{\partial r} & \frac{\partial H_3}{\partial r} & \frac{\partial H_4}{\partial r} \end{bmatrix} \begin{Bmatrix} P_1^e \\ P_2^e \\ P_3^e \\ P_4^e \end{Bmatrix}$$

$$\frac{dP}{dr} = -6 \text{ N/m}$$

$$\frac{dP}{dx} = \frac{\partial P}{\partial r} \cdot \frac{\partial r}{\partial x} = (-6 \text{ N/m})(1/10m) = -0.6 \text{ N/m}^2$$

– Force

$$F = \int_{x_1}^{x_2} P(x)\, dx = \int_0^1 P(r) \frac{dx}{dr}\, dr = \int_0^1 H(r) P^e L^e\, dr$$

$$= L^e \int_0^1 [H_1 \; H_2 \; H_3 \; H_4]\, dr \cdot P^e$$

$$\int_0^1 H_1\, dr = r + \frac{11r^2}{4} + 3r^3 - \frac{9r^4}{8} \Big|_0^1 = \frac{1}{8}$$

$$\int_0^1 H_2\, dr = \frac{9r^2}{2} - \frac{15r^3}{2} + \frac{27r^4}{8} \Big|_0^1 = \frac{3}{8}$$

$$\int_0^1 H_3\, dr = \frac{9r^2}{4} + 6r^3 - \frac{27r^4}{8} \Big|_0^1 = \frac{3}{8}$$

$$\int_0^1 H_4\, dr = \frac{r^2}{2} - \frac{3r^3}{2} + \frac{r^2}{2} \Big|_0^1 = \frac{1}{8}$$

$$F = L^e \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{Bmatrix} P_1^e \\ P_2^e \\ P_3^e \\ P_4^e \end{Bmatrix} = 10m \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{Bmatrix} 40\ N/m \\ 38\ N/m \\ 36\ N/m \\ 34\ N/m \end{Bmatrix}$$

$F = 370\ N$ ✓

- moment

$$m = \int_{x_1}^{x_2} x\ p(x)\ dx = \int_0^1 x(r)\ p(r)\ \frac{dx}{dr}\ dr = \int_0^1 (\underline{H}(r)\ x^e)^T (H(r)\ P^e) L^e\ dr$$

$$m = L^e x^{eT} \int_0^1 \underline{H}^T(r)\ \underline{H}(r)\ dr\ P^e$$

$$m = 10m \cdot \{20m \quad 23.\overline{33}m \quad 26.\overline{67}m \quad 30m\} \begin{bmatrix} \frac{8}{105} & \frac{33}{560} & \frac{-3}{140} & \frac{19}{1680} \\ \frac{33}{560} & \frac{27}{70} & \frac{-27}{560} & \frac{-3}{140} \\ \frac{-3}{140} & \frac{-27}{560} & \frac{27}{70} & \frac{33}{560} \\ \frac{19}{1680} & \frac{-3}{140} & \frac{33}{560} & \frac{8}{105} \end{bmatrix} \begin{Bmatrix} 40\ N/m \\ 38\ N/m \\ 36\ N/m \\ 34\ N/m \end{Bmatrix}$$ ✓

✓

$m = 9200\ N \cdot m$

<u>Solution #2</u>: Natural coordinate element + cubic interpolation

$$-1 \leq a \leq 1 \qquad a = -0.26 \longrightarrow r = \frac{a+1}{2}$$

$$x(a) = \sum H_i(a)\, x_i^e \quad ; \quad P(a) = \sum H_i(a)\, P_i^e$$

- Position

$$X(-0.26) \rightarrow X(r) = 23.7m$$

$$\frac{dx}{da} = \frac{d}{da} \underline{H}(a) x^e = \frac{d}{da} \underline{H}(-0.26) x^e = S_m = \frac{L^e}{2}$$

$$\frac{da}{dx} = \frac{2}{L^e}$$

- Pressure

$$P(-0.26) = P(r) = 37.78\ N/m$$

- Pressure gradient:

$$\frac{dP}{dx} = \frac{\partial P}{\partial a}\frac{\partial a}{\partial x}$$

$$\frac{\partial P}{\partial a} = \frac{\partial H(a)}{\partial a} P^e = -3\ N/m$$

$$\frac{dP}{dx} = (-3\ N/m)\left(\frac{1}{5m}\right) = -0.6\ N/m^2$$

- Force

$$F = \int_{-1}^{1} H(a)\, P^e \left(\frac{L^e}{2}\right) da = \frac{L^e}{2}\int_{-1}^{1} H(a)\, da\, P^e$$

$$F = \frac{10m}{2}\begin{bmatrix} \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} \end{bmatrix}\begin{Bmatrix} 40\ N/m \\ \\ 34\ N/m \end{Bmatrix} = 370\ N$$

- Moment

$$M = \int_{x_1}^{x_2} x\, P(x) = \int_{-1}^{1} x(a)\, P(a)\, \frac{dx}{da}\, da$$

$$M = 5m \cdot \{20m \quad 23.33m \quad 26.67m \quad 30m\}\begin{bmatrix} \frac{16}{105} & \frac{33}{280} & \frac{-3}{70} & \frac{19}{840} \\ \ddots & \frac{27}{35} & \frac{-27}{280} & \frac{-3}{70} \\ & \ddots & \frac{27}{35} & \frac{33}{280} \\ & & \ddots & \frac{16}{105} \end{bmatrix}\begin{Bmatrix} 40\ N/m \\ \\ 34\ N/m \end{Bmatrix}$$

$$M = 9200\ N \cdot m$$

Homework 7

Solution 1

$$H1 := \frac{2 - 11\,r + 18\,r^2 - 9\,r^3}{2}$$

$$H2 := \frac{18\,r - 45\,r^2 + 27\,r^3}{2}$$

$$H3 := \frac{-9\,r + 36\,r^2 - 27\,r^3}{2}$$

$$H4 := \frac{2\,r - 9\,r^2 + 9\,r^3}{2}$$

$$H := \begin{bmatrix} H1 & H2 & H3 & H4 \end{bmatrix}$$

$$x_e := \begin{bmatrix} 20 \\ 23.33 \\ 26.67 \\ 30 \end{bmatrix} \qquad P_e := \begin{bmatrix} 40 \\ 38 \\ 36 \\ 34 \end{bmatrix}$$

$$dH1 := \frac{-11 + 36\,r - 27\,r^2}{2}$$

$$dH2 := \frac{18 - 90\,r + 81\,r^2}{2}$$

$$dH3 := \frac{-9 + 72\,r - 81\,r^2}{2}$$

$$dH4 := \frac{2 - 18\,r + 27\,r^2}{2}$$

$$dH := \begin{bmatrix} dH1 & dH2 & dH3 & dH4 \end{bmatrix}$$

$r := 0.37$

$H1 := \dfrac{2 - 11\ r + 18\ r^2 - 9\ r^3}{2}$

$H2 := \dfrac{18\ r - 45\ r^2 + 27\ r^3}{2}$

$H3 := \dfrac{-9\ r + 36\ r^2 - 27\ r^3}{2}$

$H4 := \dfrac{2\ r - 9\ r^2 + 9\ r^3}{2}$

$dH1 := \dfrac{-11 + 36\ r - 27\ r^2}{2}$

$dH2 := \dfrac{18 - 90\ r + 81\ r^2}{2}$

$dH3 := \dfrac{-9 + 72\ r - 81\ r^2}{2}$

$dH4 := \dfrac{2 - 18\ r + 27\ r^2}{2}$

$H := [H1\ \ H2\ \ H3\ \ H4]$

$dH := [dH1\ \ dH2\ \ dH3\ \ dH4]$

$H \cdot x_e = 23.697$

$H \cdot P_e = 37.78$

$dH \cdot x_e = 10.018$

$dH \cdot P_e = -6$

Solution #2

$r := \left(\dfrac{a + 1}{2}\right)$

$H1 := \dfrac{2 - 11\ r + 18\ r^2 - 9\ r^3}{2}$

$H2 := \dfrac{18\ r - 45\ r^2 + 27\ r^3}{2}$

$H3 := \dfrac{-9\ r + 36\ r^2 - 27\ r^3}{2}$

$H4 := \dfrac{2\ r - 9\ r^2 + 9\ r^3}{2}$

$H := [H1\ \ H2\ \ H3\ \ H4]$

$$dH1 := \frac{d}{da}\left(\frac{2-11\ r+18\ r^2-9\ r^3}{2}\right) \rightarrow \frac{9 \cdot a}{2} - \frac{27 \cdot \left(\frac{a}{2}+\frac{1}{2}\right)^2}{4} + \frac{7}{4}$$

$$dH2 := \frac{d}{da}\left(\frac{18\ r-45\ r^2+27\ r^3}{2}\right) \rightarrow \frac{81 \cdot \left(\frac{a}{2}+\frac{1}{2}\right)^2}{4} - \frac{45 \cdot a}{4} - \frac{27}{4}$$

$$dH3 := \frac{d}{da}\left(\frac{-9\ r+36\ r^2-27\ r^3}{2}\right) \rightarrow 9 \cdot a - \frac{81 \cdot \left(\frac{a}{2}+\frac{1}{2}\right)^2}{4} + \frac{27}{4}$$

$$dH4 := \frac{d}{da}\left(\frac{2\ r-9\ r^2+9\ r^3}{2}\right) \rightarrow \frac{27 \cdot \left(\frac{a}{2}+\frac{1}{2}\right)^2}{4} - \frac{9 \cdot a}{4} - \frac{7}{4}$$

$$int1 := \int_{-1}^{1} H1\ da \rightarrow \frac{1}{4}$$

$$int2 := \int_{-1}^{1} H2\ da \rightarrow \frac{3}{4}$$

$$int3 := \int_{-1}^{1} H3\ da \rightarrow \frac{3}{4}$$

$$int4 := \int_{-1}^{1} H4\ da \rightarrow \frac{1}{4}$$

$$intH := [int1\ \ int2\ \ int3\ \ int4]$$

$$5 \cdot intH \cdot P_e = 370$$

$$HH_{int} := \int_{-1}^{1} \left(H^T \cdot H\right) da \rightarrow \begin{bmatrix} \frac{16}{105} & \frac{33}{280} & -\frac{3}{70} & \frac{19}{840} \\ \frac{33}{280} & \frac{27}{35} & -\frac{27}{280} & -\frac{3}{70} \\ -\frac{3}{70} & -\frac{27}{280} & \frac{27}{35} & \frac{33}{280} \\ \frac{19}{840} & -\frac{3}{70} & \frac{33}{280} & \frac{16}{105} \end{bmatrix}$$

$$5 \cdot {x_e}^T \cdot HH_{int} \cdot P_e = 9.2 \cdot 10^3$$

$a := -0.26$ $\qquad r := \left(\frac{a+1}{2}\right) = 0.37$

$$H1 := \frac{2 - 11\ r + 18\ r^2 - 9\ r^3}{2}$$

$$H2 := \frac{18\ r - 45\ r^2 + 27\ r^3}{2}$$

$$H3 := \frac{-9\ r + 36\ r^2 - 27\ r^3}{2}$$

$$H4 := \frac{2\ r - 9\ r^2 + 9\ r^3}{2}$$

$$H := [H1\ \ H2\ \ H3\ \ H4]$$

$$H \cdot x_e = 23.697$$

$$dH1 := \frac{9 \cdot a}{2} - \frac{27 \cdot \left(\frac{a}{2} + \frac{1}{2}\right)^2}{4} + \frac{7}{4}$$

$$dH2 := \frac{81 \cdot \left(\frac{a}{2} + \frac{1}{2}\right)^2}{4} - \frac{45 \cdot a}{4} - \frac{27}{4}$$

$$dH3 := 9 \cdot a - \frac{81 \cdot \left(\frac{a}{2} + \frac{1}{2}\right)^2}{4} + \frac{27}{4}$$

$$dH4 := \frac{27 \cdot \left(\frac{a}{2} + \frac{1}{2}\right)^2}{4} - \frac{9 \cdot a}{4} - \frac{7}{4}$$

$$dH := [dH1\ \ dH2\ \ dH3\ \ dH4]$$

$$dH \cdot x_e = 5.009$$

$$dH \cdot P_e = -3$$

$$int1 := \int H1\,\mathrm{d}r \rightarrow 3\cdot r^3 - \frac{9\cdot r^4}{8} - \frac{11\cdot r^2}{4} + r$$

$$int2 := \int H2\,\mathrm{d}r \rightarrow \frac{27\cdot r^4}{8} - \frac{15\cdot r^3}{2} + \frac{9\cdot r^2}{2}$$

$$int3 := \int H3\,\mathrm{d}r \rightarrow 6\cdot r^3 - \frac{27\cdot r^4}{8} - \frac{9\cdot r^2}{4}$$

$$int4 := \int H4\,\mathrm{d}r \rightarrow \frac{9\cdot r^4}{8} - \frac{3\cdot r^3}{2} + \frac{r^2}{2}$$

$$int1 := \int_0^1 H1\,\mathrm{d}r \rightarrow \frac{1}{8}$$

$$int2 := \int_0^1 H2\,\mathrm{d}r \rightarrow \frac{3}{8}$$

$$int3 := \int_0^1 H3\,\mathrm{d}r \rightarrow \frac{3}{8}$$

$$int4 := \int_0^1 H4\,\mathrm{d}r \rightarrow \frac{1}{8}$$

$$HH_{int} := \int_0^1 \left(H^{\mathrm{T}}\cdot H\right)\mathrm{d}r \rightarrow \begin{bmatrix} \frac{8}{105} & \frac{33}{560} & -\frac{3}{140} & \frac{19}{1680} \\ \frac{33}{560} & \frac{27}{70} & -\frac{27}{560} & -\frac{3}{140} \\ -\frac{3}{140} & -\frac{27}{560} & \frac{27}{70} & \frac{33}{560} \\ \frac{19}{1680} & -\frac{3}{140} & \frac{33}{560} & \frac{8}{105} \end{bmatrix}$$

$$intH := \begin{bmatrix} int1 & int2 & int3 & int4 \end{bmatrix}$$

$$10\cdot intH\cdot P_e = 370$$

$$10\cdot {x_e}^{\mathrm{T}}\cdot HH_{int}\cdot P_e = 9.2\cdot 10^3$$