MECH 417 - Homework 7

Problem 1:

Assuming that the geometry of the problem does not change, then the node position vector x^e and pressure value at each node p^e are given by

$$x^e = \begin{bmatrix} 20 \\ 25 \\ 30 \end{bmatrix} \text{ m}, \quad p^e = \begin{bmatrix} 40 \\ 37 \\ 34 \end{bmatrix} \text{ N/m}$$

Additionally, the 1D quadratic interpolation functions in unit coordinate space are

$$H(r) = \begin{bmatrix} 1 - 3r + 2r^2, & 4r - 4r^2, & 2r^2 - r \end{bmatrix}$$

The map which takes the real space to the unit coordinate space is

$$r(x) = \frac{1}{10}(x - 20),$$

so the inverse of the Jacobian of this transformation is

$$\frac{dx}{dr} = \left(\frac{dr}{dx}\right)^{-1} = \left(\frac{1}{10}\text{ m}^{-1}\right)^{-1} = 10 \text{ m}.$$

Therefore the force and moment can be computed:

$$F = \int_{x_1}^{x_2} p(x)dx = \int_0^1 p(r)\frac{dx}{dr}dr = (10\text{ m})\int_0^1 H(r)p^e dr = (10\text{ m})\int_0^1 H(r)dr\; p^e$$

$$= (10\text{ m})\int_0^1 \begin{bmatrix} 1 - 3r + 2r^2, & 4r - 4r^2, & 2r^2 - 2r \end{bmatrix} dr \begin{bmatrix} 40 \\ 37 \\ 34 \end{bmatrix} \text{ N/m}$$

$$= (10\text{ m}) \begin{bmatrix} r - \frac{3}{2}r^2 + \frac{2}{3}r^3, & 2r^2 - \frac{4}{3}r^3, & \frac{2}{3}r^3 - \frac{1}{2}r^2 \end{bmatrix}_0^1 \begin{bmatrix} 40 \\ 37 \\ 34 \end{bmatrix} \text{ N/m}$$

$$= (10\text{ m}) \begin{bmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 40 \\ 37 \\ 34 \end{bmatrix} \text{ N/m} = (10\text{ m})\left(\frac{40}{6} + \frac{74}{3} + \frac{34}{6}\right) \text{ N/m} = 370 \text{ N}$$

$$M = \int_{x_1}^{x_2} xp(x)dx = \int_0^1 (H(r)x^e)^T (H(r)p^e)\frac{dx}{dr}dr = (10\text{ m})(x^e)^T \int_0^1 H(r)^T H(r)dr\ p^e$$

$$= (10\text{ m})\left(\begin{bmatrix}20 & 25 & 30\end{bmatrix}\text{ m}\right)\int_0^1 H(r)^T H(r)dr\left(\begin{bmatrix}40\\37\\34\end{bmatrix}\text{ N/m}\right)$$

Calculating the integral by itself:

$$\int_0^1 H(r)^T H(r)dr = \int_0^1 \begin{bmatrix}(1-3r+2r^2)^2 & (4r-4r^2)(1-3r+2r^2) & (2r^2-r)(1-3r+2r^2)\\ (1-3r+2r^2)(4r-4r^2) & (4r-4r^2)^2 & (2r^2-r)(4r-4r^2)\\ (1-3r+2r^2)(2r^2-2) & (4r-4r^2)(2r^2-r) & (2r^2-2)^2\end{bmatrix} dr$$

$$= \begin{bmatrix}\frac{2}{15} & \frac{1}{15} & \frac{-1}{30}\\ \frac{1}{15} & \frac{8}{15} & \frac{1}{15}\\ \frac{-1}{30} & \frac{1}{15} & \frac{2}{15}\end{bmatrix}$$

Therefore the moment is

$$\begin{aligned}
M &= (10\text{ m})\left(\begin{bmatrix}20 & 25 & 30\end{bmatrix}\text{m}\right)\begin{bmatrix}\frac{2}{15} & \frac{1}{15} & \frac{-1}{30}\\ \frac{1}{15} & \frac{8}{15} & \frac{1}{15}\\ \frac{-1}{30} & \frac{1}{15} & \frac{2}{15}\end{bmatrix}\left(\begin{bmatrix}40\\37\\34\end{bmatrix}\text{ N/m}\right)\\
&= (10\text{ m})\left(\begin{bmatrix}20 & 25 & 30\end{bmatrix}\text{m}\right)\left(\begin{bmatrix}\frac{20}{3}\\ \frac{74}{3}\\ \frac{17}{3}\end{bmatrix}\text{ N/m}\right)\\
&= (10\text{ m})(920\text{ N})\\
&= 9200\text{ N-m.}
\end{aligned}$$

MECH 517 Homework 7

Problem 1:
Repeat the solution for the parametric line pressure example using a cubic pressure interpolation. After the integrals are complete set the internal node pressures to linear from p_1 and p_2 before substituting in the numerical values.

Solution:
Setting the internal node pressures to linear from p_1 to p_2 gives:

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 38 \\ 36 \\ 34 \end{bmatrix} N/m$$

Parameterizing the position and pressure in terms of a unit coordinate element, and using a cubic interpolation function for the pressure, gives:

$$x(r) = \sum_{j=1}^{2} H_j(r) x_j^e = (1-r)x_1^e + r x_2^e$$

$$p(r) = H(r)p^e = (1 - \frac{11}{2}r + 9r^2 - \frac{9r^3}{2})p_1^e + (9r - \frac{45r^2}{2} + \frac{27r^3}{2})p_2^e + (\frac{-9r}{2} + 18r^2 - \frac{27r^3}{2})p_3^e + (r - \frac{9r^2}{2} + \frac{9r^3}{2})p_4^e;$$

At the local point of interest, $r = 0.37$, we obtain the following values:

$$x(0.37) = \sum_{j=1}^{2} H_j(r) x_j^e = (1-0.37)(20m)^e + (0.37)(30m)^e$$

$$\mathbf{x(0.37) = 23.7m}$$

$$p(0.37) = H(r)p^e = (1 - \frac{11}{2}(0.37) + 9(0.37)^2 - \frac{9(0.37)^3}{2})(40N/m)^e + (9(0.37) - \frac{45(0.37)^2}{2} + \frac{27(0.37)^3}{2})(38N/m)^e + (\frac{-9(0.37)}{2} + 18(0.37)^2 - \frac{27(0.37)^3}{2})(36)^e + ((0.37) - \frac{9(0.37)^2}{2} + \frac{9(0.37)^3}{2})(34)^e;$$

$$\mathbf{p(0.37) = 37.78N/m}$$

The pressure gradient is calculated as follows:

$$\frac{dp}{dx} = \frac{dp}{dr}\frac{dr}{dx}$$

$$\frac{dx}{dr} = \frac{d}{dr}H(r)x^e = (-1)x_1^e + (1)x_2^e = (x_2 - x_1)^e = L^e = 10m$$

$$\frac{dr}{dx} = \frac{1}{L^e} = \frac{1}{10m}$$

$$\frac{dp}{dr} = \frac{d}{dr}H(r)p^e = (-\frac{11}{2} + 18(0.37) - \frac{27(0.37)^2}{2})(40N/m)^e +$$

$$(9 - 45(0.37) + \frac{81(0.37)^2}{2})(38N/m)^e +$$

$$(\frac{-9}{2} + 36(0.37) - \frac{81(0.37)^2}{2})(36)^e +$$

$$(1 - 9(0.37) + \frac{27(0.37)^2}{2})(34)^e;$$

$$= -6N/m$$

$$\frac{dp}{dx} = (-6N/m)(\frac{1}{10m}) = \mathbf{-0.6N/m^2}$$

The total force due to the pressure can be found by integrating the pressure across the length of the bar as follows. For clarity, the interpolation functions used for length and pressure are shown below:

$$H(r_x) = [(1-r) \quad r]$$

$$H(r_p) = [(1 - \frac{11}{2}r + 9r^2 - \frac{9r^3}{2}) \quad (9r - \frac{45r^2}{2} + \frac{27r^3}{2}) \ldots$$
$$(\frac{-9r}{2} + 18r^2 - \frac{27r^3}{2}) \quad (r - \frac{9r^2}{2} + \frac{9r^3}{2})]$$

$$F = \int_{x_1}^{x_2} p(x)dx = \int_0^1 p(r)\frac{dx}{dr}dr$$

$$F = \int_0^1 H(r_p)p^e L^e dr$$

$$= L^e \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 40 \\ 38 \\ 36 \\ 34 \end{bmatrix}$$

$$F = \frac{10}{8}(40 + 3(38) + 3(36) + 34) = \mathbf{370N}$$

The moment can be computed by integrating the length multiplied by the pressure over the length of the bar as shown below.

$$M = \int_{x_1}^{x_2} xp(x) = \int_0^1 x(r)p(r)\frac{dx}{dr}dr$$

$$M = \int_0^1 (H(r_x)x^e)^T(H(r_p)p^e)L^e dr$$

$$M = L^e x^{eT} \int_0^1 H(r_x)^T H(r_p)dr$$

$$= (10m)\begin{bmatrix}20m & 30m\end{bmatrix}\begin{bmatrix}\frac{13}{120} & \frac{3}{10} & \frac{3}{40} & \frac{1}{60} \\ \frac{1}{60} & \frac{3}{40} & \frac{3}{10} & \frac{13}{120}\end{bmatrix}\begin{bmatrix}40\\38\\36\\34\end{bmatrix} N/m$$

$\mathbf{M = 9200\ N - m}$

The MATLAB code used to derive these values has been appended on the next page.